# Finite-dimensional global attractor for a nonlocal phase-field system *

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### Abstract

We analyze a phase-field system where the energy balance equation is linearly coupled with a nonlinear and nonlocal ODE for the order parameter $\chi$. The latter equation is characterized by a space convolution term which models particle interaction and a singular configuration potential that forces $\chi$ to take values in $(-1, 1)$. We prove that the corresponding dynamical system has a bounded absorbing set in a suitable phase space. Then we establish the existence of a finite-dimensional global attractor.

**Key words:** phase-field models, singular potentials, nonlocal operators, finite dimensional global attractors.

**AMS (MOS) subject classification:** 35B41, 35Q99, 80A22.

## 1 Introduction

Consider a two-phase material (like, for instance, a mixture of ice and water) which occupies a bounded domain in $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$ and denote by $\vartheta$ its relative temperature with respect to a given (constant) critical one (e.g., the one at which the two phases coexist). A well-known model which accounts for phase changes due to the variation of $\vartheta$ only was proposed and analyzed in [13] (see also [12] and references therein). This is based on the Ginzburg-Landau theory of phase transitions and it assumes as further variable an order parameter (or phase-field) $\chi$ which characterizes, for instance, the most energetic phase of the material (say water, in a water-ice system).

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Using a phenomenological argument, it is postulated that the evolution of $\chi$ is ruled by a *gradient flow* of the form

$$\chi_t = -\delta_\chi E(\vartheta, \chi), \tag{1.1}$$

$\delta_\chi$ being the variational derivative with respect to $\chi$ of the *free energy $E$* defined by

$$E(\vartheta, \chi) = \int_\Omega \left( \frac{\nu}{2} |\nabla \chi(x)|^2 + W(\chi(x)) - \alpha \vartheta(x) \chi(x) \right) \, \mathrm{d}x, \tag{1.2}$$

where $\nu > 0$ and $\alpha \in \mathbb{R}$ are given constants, the latter one being related to the latent heat. Here $W$ is the (density of) potential energy associated with the phase configuration which can be defined either on a finite interval (e.g. $(-1, 1)$) or on the whole real line. In the first case, $W$ is called *singular* and the most typical form is the following

$$W(r) = (1 + r)\ln(1 + r) + (1 - r)\ln(1 - r) - \frac{\lambda}{2}r^2, \quad r \in (-1, 1), \tag{1.3}$$

where $\lambda \in \mathbb{R}$. However, though singular potentials are more acceptable from the modeling viewpoint, $W$ is very often approximated by a polynomial function defined on $\mathbb{R}$ like, e.g., $W(r) = (r^2 - 1)^2$. In this case the potential is called *smooth*.

It is worth observing that the term on the right hand side of (1.1) may be viewed as a sort of *generalized force* driving the evolution of $\chi$, that is, the phase transformation. Combining now the balance equation for the internal energy density with (1.1) we find the evolution system

$$(\vartheta + \alpha \chi)_t - \Delta \vartheta = f, \tag{1.4}$$
$$\chi_t - \nu \Delta \chi + W'(\chi) - \alpha \vartheta = 0, \tag{1.5}$$

in $\Omega \times (0, +\infty)$, where $f$ represents a given (volumic) heat source and some constants have been taken equal to one for simplicity. Equations (1.4)-(1.5) are also known as Caginalp phase-field system. An important feature of this rather simple model is the fact that its (formal) asymptotic limits are well-known sharp interface problems (see, e.g., [14, 15, 17]). We recall that phase-field systems are diffuse interface models in the sense that the sharp interface separating two different phases, which is usually very difficult to handle theoretically and numerically, is replaced by the level set of a suitable (smooth) order parameter.

From the mathematical viewpoint, system (1.4)-(1.5) endowed with initial conditions and various types of boundary conditions has been investigated by many authors (see, e.g., [9, 10, 11, 18, 21, 22, 23, 25, 33, 34, 38, 39, 40, 46, 50, 51, 54] and references therein), even under more general assumptions (for instance, where $\alpha$ is a function depending on $\chi$). Besides well-posedness results, an important issue is the longtime behavior of solutions. This behavior is usually non trivial since the set of stationary states is a *continuum* if the spatial dimension is greater than one. Thus the existence of a Lyapunov functional does not guarantee that a given trajectory converges to a single equilibrium. To prove that the Łojasiewicz-Simon approach has been employed (see, in particular, [18, 21, 22, 34, 54]). Regarding the global dynamics, system (1.4)-(1.5), endowed with appropriate boundary conditions, can be interpreted as a dynamical system

in a suitable phase space. This system is dissipative, i.e., there exists a bounded absorbing set in the phase space. In addition, it possesses a finite-dimensional global attractor as well as an exponential attractors (cf., for instance, [9, 10, 11, 18, 33, 39, 40, 46], see also [30, 47] for related results).

On the other hand, the free energy $E$ can be viewed as an approximation of a nonlocal expression of the following type (see [19] and references therein, cf. also [28, 29, 31, 32] for phase separation models)

$$E_{\text{nl}}(\vartheta, \chi) = \int_{\Omega} \left( \int_{\Omega} k(x-y) \frac{|\chi(x) - \chi(y)|^2}{4} \, \mathrm{d}y + W(\chi(x)) - \alpha\vartheta(x)\chi(x) \right) \, \mathrm{d}x, \quad (1.6)$$

where $k : \mathbb{R}^d \to \mathbb{R}$ is an interaction kernel satisfying $k(x) = k(-x)$ and such that

$$\kappa(x) := \int_{\Omega} k(x-y) \, \mathrm{d}y \tag{1.7}$$

is bounded and nonnegative (see, for instance, [29, Rem. 2.2] for concrete examples). Indeed one can recover (1.2) by taking

$$\int_{\Omega} \frac{\nu}{2} |\nabla \chi(x)|^2 \, \mathrm{d}x,$$

as a first approximation of

$$\int_{\Omega} \int_{\Omega} k(x-y) |\chi(x) - \chi(y)|^2 \, \mathrm{d}y \, \mathrm{d}x,$$

where $\nu = 2 \int_{\Omega} k(y)(y_i)^2 dy$ is supposed to be independent of coordinate $i$. In concrete examples $k$ is localized in a neighborhood of 0 so that $\nu$ is related to the thickness of the interface between the components. It is interesting to point out that this kind of approximation was already introduced by Van der Waals in his celebrated paper [48] and since then it was widely adopted in the mathematical literature on phase transitions. This might be due to the fact that most people are more used to deal with PDE rather than nonlocal operators.

The evolution system which derives from the nonlocal free energy (1.6) takes the form

$$\vartheta_t + \alpha\chi_t - \Delta\vartheta = f, \tag{1.8}$$
$$\chi_t + \kappa\chi + W'(\chi) = \mathcal{J}[\chi] + \alpha\vartheta, \tag{1.9}$$

in $\Omega \times (0, +\infty)$, where

$$\mathcal{J}[v](x) = \int_{\Omega} k(x-y)v(y) \, \mathrm{d}y, \qquad x \in \Omega. \tag{1.10}$$

The well-posedness for this system was firstly established in [5] through a semigroup approach in the case $\Omega = \mathbb{R}$ for a smooth $W$. In a similar setting, existence of traveling waves for small $\alpha$ was proven in [3] (see also [2, 4, 6, 7, 27, 53] for results on the nonlocal Allen-Cahn equation, i.e., (1.9) with $\alpha = 0$). Nucleation simulation by using

nonlocal interactions has been studied in [49]. Phase interface dynamics associated with (1.8)-(1.9) was analyzed in [19] and [20] (cf. also references therein) by formal asymptotics (cf. also [16] for higher-order approximations and [24] for a related model). The case of bounded multi-dimensional domains was examined in [8] and [26] (cf. also [42, 43, 44, 45, 52] for results on more refined models and [1] for the numerical analysis). Well-posedness issues were discussed in [8] when $\vartheta$ is subject to homogeneous Neumann boundary condition and $W$ is smooth. In addition, some results concerned with the asymptotic behavior were demonstrated (e.g., the existence of a bounded absorbing set in one spatial dimension). In [26], the potential $W$ is smooth as well, but $\vartheta$ is subject to Dirichlet homogeneous. There the main goal was to establish the convergence of a given trajectory to a single stationary state an this fact was proven by means of a generalized version of the Łojasiewicz-Simon inequality, provided that $W$ is real analytic. Note that, also in the nonlocal case, the set of stationary states can be a *continuum* in dimension greater than one, even though $z \mapsto W'(z) + \kappa z$ is invertible (see [26, Introduction], cf. also [6] for the nonlocal Allen-Cahn equation). A similar issue is analyzed in the more recent contribution [35], where equation (1.9) has an additional relaxation term of the form $\varepsilon \chi_{tt}$ and $W$ is a singular potential (though not of type (1.3)).

However, none of the above results is concerned with the existence of a global attractor even though this is a rather natural feature of phase-field dynamics (cf. above for the local models). A possible explanation might be that it is not evident how to show some precompactness for the variable $\chi$. In this note we show how the existence of a finite-dimensional global attractor can be established. This is achieved by estimating the differences of two trajectories which originated from a suitable bounded absorbing set. Such a difference is controlled by a contractive part depending on the initial data plus a precompact term. For the latter it is essential (but *natural*) to assume some compactness on the linear nonlocal operator $\mathcal{J}$. For the sake of simplicity, we will take full advantage of the setting developed in [35]. Thus we will suppose that $\vartheta$ satisfies Dirichlet homogeneous boundary condition, while $W(r)$ will be a singular potential defined on $(-1, 1)$ which goes to $+\infty$ as $r$ approaches the pure states $\pm 1$. This will allow us to easily deduce well-posedness since the model studied in [35] essentially reduces to ours when we take $\varepsilon = 0$. Nonetheless, we think that our argument can also be applied to different boundary conditions for $\vartheta$ (e.g., homogeneous Neumann) and, especially, to other types of potentials like, for instance, (1.3) (see Remarks 2.6 and 4.4 below).

The paper is organized as follows. In the next section we discuss well-posedness issues. Section 3 contains a basic uniform in time estimate which is exploited in Section 4 to show that our problem defines a dissipative dynamical system on a suitable phase space. Then we conclude with our main result: the existence of a finite-dimensional global attractor.

## 2  Well-posedness

Let us introduce some notation first. Set $H := L^2(\Omega)$ and $V := H_0^1(\Omega)$ and denote by $(\cdot, \cdot)$ the scalar product in $H$ and by $\| \cdot \|$ the induced norm. In general, $\| \cdot \|_X$

will indicate the norm in a generic real Banach space $X$. Let $V$ be endowed with the norm $\| \cdot \|_V := \| \nabla \cdot \|$. Let us identify $H$ with its topological dual $H'$ so that we have the continuous and compact inclusions $V \subset H \subset V'$. Moreover, we set $A := -\Delta : D(A) = H^2(\Omega) \cap V \to H$.

Regarding the potential $W$, we suppose

$$W \in C^2((-1,1); \mathbb{R}^+), \qquad \lim_{r \to 1^-, -1^+} W(r) = +\infty, \tag{2.1}$$

$$\exists \, \lambda \in \mathbb{R} : \quad W''(r) \geq -\lambda, \qquad \forall \, r \in (-1,1). \tag{2.2}$$

It is easy to see that (2.1)–(2.2) entail

$$W'(r)r \geq W(r) - \frac{\lambda r^2}{2} - W(0), \quad \forall \, r \in (-1,1). \tag{2.3}$$

Moreover, if $v \in L^\infty(\Omega)$ is any function such that $W(v) \in L^\infty(\Omega)$, then

$$\exists \, \delta = \delta\big(W, \|v\|_{L^\infty(\Omega)}\big) \in (0,1) \quad \text{such that} \quad -1 + \delta \leq v(x) \leq 1 - \delta \quad \text{for a.e. } x \in \Omega. \tag{2.4}$$

Consider now $\kappa$ defined by (1.7) and the nonlocal operator $\mathcal{J}$ (cf. (1.10)). We require (cf., for instance, [29])

$$\int_\Omega \int_\Omega |k(x-y)| dy dx = k_0 < +\infty, \tag{J1}$$

$$\operatorname{ess\,sup}_{x \in \Omega} \int_\Omega |k(x-y)| \, \mathrm{d}y = k_1 < +\infty. \tag{J2}$$

Observe that the above assumptions entail that $\mathcal{J} \in \mathcal{L}(H; H)$ is self-adjoint and compact. Also, $\mathcal{J}$ is compact from $L^\infty(\Omega)$ to $C(\overline{\Omega})$. System (1.8)–(1.9) with $\vartheta$ subject to Dirichlet homogeneous boundary condition and initial conditions can now be written as follows

$$\vartheta_t + \alpha \chi_t + A\vartheta = f, \qquad \text{a.e. in } \Omega \times (0,T), \tag{2.5}$$

$$\chi_t + \kappa \chi + W'(\chi) = \mathcal{J}[\chi] + \alpha \vartheta, \qquad \text{a.e. in } \Omega \times (0,T), \tag{2.6}$$

$$\vartheta_{|t=0} = \vartheta_0, \quad \chi_{|t=0} = \chi_0 \qquad \text{a.e. in } \Omega, \tag{2.7}$$

where

$$\vartheta_0 \in V, \tag{2.8}$$

and

$$f \in H. \tag{2.9}$$

Moreover, we suppose that

$$\chi_0 \in L^\infty(\Omega) \quad \text{s.t. } \exists \, \delta_0 \in (0,1) : \quad -1 + \delta_0 \leq \chi_0 \leq 1 - \delta_0 \quad \text{a.e. in } \Omega, \tag{2.10}$$

where the latter property is equivalent to say that $W(\chi_0) \in L^\infty(\Omega)$.

Well-posedness can be proven arguing as in the proof of [35, Thm.2.1] with $\varepsilon = 0$ (cf. also [35, Rem.2.7])

**Theorem 2.1.** *Let* (2.1)–(2.2), (J1)–(J2), (2.8)–(2.10) *hold. Then, for any given* $T > 0$, *there exists one and only one pair* $(\vartheta, \chi)$ *such that*

$$\vartheta \in L^2(0, T; D(A)) \cap H^1(0, T; H) \cap C^0([0, T]; V), \tag{2.11}$$

$$\chi, \, W(\chi) \in L^\infty(0, T; L^\infty(\Omega)), \quad \chi_t \in L^2(0, T; L^\infty(\Omega)), \tag{2.12}$$

*which solves* (2.5)-(2.7). *Moreover, there exists* $\delta = \delta(W, k_0, \kappa, \alpha, f, \delta_0) \in (0, 1)$ *such that*

$$-1 + \delta \le \chi(t) \le 1 - \delta \quad \text{a.e. in } \Omega,$$

*for almost any* $t \in (0, T)$. *Next, given two triplets* $(\vartheta_{0,1}, \chi_{0,1})$, $(\vartheta_{0,2}, \chi_{0,2})$ *of initial data satisfying conditions* (2.8)–(2.10) *(the latter w.r.t. possibly different constants* $\delta_{0i} > 0$, $i = 1, 2$) *and denoting the corresponding solutions by* $(\vartheta_1, \chi_1)$, $(\vartheta_2, \chi_2)$, *respectively, we have the continuous dependence estimate*

$$\|(\vartheta_1 - \vartheta_2)(t)\| + \|\nabla(\vartheta_1 - \vartheta_2)\|_{L^2(0,t;H)} + \|(\chi_1 - \chi_2)(t)\|$$
$$\le \Lambda_0 \big( \|\vartheta_{0,1} - \vartheta_{0,2}\| + \|\chi_{0,1} - \chi_{0,2}\| \big), \quad \forall t \in [0, T], \tag{2.13}$$

*where the positive constant* $\Lambda_0$ *depends on* $T$, $\Omega$, $W$, $k_0$, $k_1$, $\alpha$, $f$, *and on the initial data (in particular, on* $\delta_{0i}$, $i = 1, 2$).

**Remark 2.2.** To establish existence instead of the fixed-point technique used in [35] one can use a vanishing viscosity argument like in [26].

We also have a higher-order control for the temperature difference, namely,

**Corollary 2.3.** *Let the assumptions of Theorem 2.1 hold. In addition to estimate* (2.13) *we have*

$$\|\nabla(\vartheta_1 - \vartheta_2)(t)\| + \|A(\vartheta_1 - \vartheta_2)\|_{L^2(0,t;H)} \le \Lambda_1 \big( \|\vartheta_{0,1} - \vartheta_{0,2}\|_V + \|\chi_{0,1} - \chi_{0,2}\| \big), \tag{2.14}$$

*for all* $t \in [0, T]$ *where* $\Lambda_1$ *is a positive constant similar to* $\Lambda_0$.

PROOF.    It suffices to write equation (2.5) for $\vartheta_1 - \vartheta_2$, multiplying by $A(\vartheta_1 - \vartheta_2)$ recalling (2.6) and using (2.13). ∎

**Remark 2.4.** Assumptions (J1)-(J2) are satisfied by concrete examples of interaction kernels like the ones mentioned in [29, Rem. 2.2].

**Remark 2.5.** Assumptions (2.8) and (2.9) can be weakened as follows

$$\vartheta \in D(A^{\frac{\rho}{2}}), \quad f \in D(A^{(\rho-1)/2}), \tag{2.15}$$

where $\rho \in (\frac{3}{4}, 1)$. In this case, we recall that $D(A^\varrho) \subset H^{2\varrho}(\Omega) \hookrightarrow L^\infty(\Omega)$ since the spatial dimension is three at most. In this case estimate (2.14) becomes

$$\|A^{\frac{\rho}{2}}(\vartheta_1 - \vartheta_2)(t)\| + \|A^{\frac{\rho+1}{2}}(\vartheta_1 - \vartheta_2)\|_{L^2(0,t;H)}$$
$$\le \Lambda_1 \big( \|A^{\frac{\rho}{2}}(\vartheta_{0,1} - \vartheta_{0,2})\| + \|\chi_{0,1} - \chi_{0,2}\| \big), \quad \forall t \in [0, T].$$

**Remark 2.6.** The second assumption (2.1) does not cover, for instance, the case of a potential which is bounded in $[-1, 1]$ like the logarithmic potential (1.3). This important class of potentials can still be handled if we deal with the Allen-Cahn equation only and temperature is assumed to be given (see Remark 4.4 below). However, system (1.8)-(1.9) with more general potentials require further arguments and will be analyzed elsewhere (see [36]).

# 3 A dissipative estimate

Here we establish some uniform in time estimates which are essentially contained in the proof of [35, Thm.2.1] taking $\varepsilon = 0$.

**Theorem 3.1.** *Let the assumptions of Theorem 2.1 hold. Then the unique solution $(\vartheta, \chi)$ to (2.5)-(2.7) satisfies the following estimate*

$$\|\vartheta(t)\|_V^2 + \|W(\cdot, \chi(t))\|_{L^\infty(\Omega)} \le C_0(1 + c_{\delta_0} + \|\vartheta_0\|_V^2)e^{-\beta t} + C_1, \qquad (3.1)$$

*for any $t \ge 0$. Here $\beta$, $C_0$ and $C_1$ are positive constants which depend on $\Omega$, $\lambda$, $W(0)$, $k_1$, $\alpha$ and $f$ at most, while $c_{\delta_0} > 0$ depends on $\delta_0$ and $W$.*

PROOF. Let us multiply equation (2.5) by $\vartheta(t) + \xi A\vartheta(t)$ for some given $\xi > 0$ to be chosen in the sequel. Integrating over $\Omega$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\vartheta\|^2 + \frac{\xi}{2}\|\nabla\vartheta\|^2\right) + \|\nabla\vartheta\|^2 + \xi\|A\vartheta\|^2 = (f - \alpha\chi_t, \vartheta + \xi A\vartheta).$$

Consider now equation (2.6). Multiplying it by $\chi_t(t) + \eta\chi(t)$, where $\eta > 0$ will be fixed in the sequel, and integrating over $\Omega$ we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta}{2}\|\chi\|^2 + \frac{1}{2}(\kappa\chi, \chi) + (W(\chi), 1)\right) + \|\chi_t\|^2 + \eta(\kappa\chi, \chi) + \eta(W'(\chi), \chi)$$
$$= (\mathcal{J}[\chi] + \alpha\vartheta, \chi_t + \eta\chi).$$

Adding the two identities we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \|\nabla\vartheta\|^2 + \xi\|A\vartheta\|^2 + \|\chi_t\|^2 + \eta(\kappa\chi, \chi) + \eta(W'(\chi), \chi)$$
$$= (f, \vartheta + \xi A\vartheta) - \xi\alpha(\chi_t, A\vartheta) + (\mathcal{J}[\chi], \chi_t + \eta\chi) + \eta\alpha(\vartheta, \chi),$$

where, for all $t \ge 0$,

$$\mathcal{E}(t) = \frac{1}{2}\|\vartheta(t)\|^2 + \frac{\xi}{2}\|\nabla\vartheta(t)\|^2 + \frac{\eta}{2}\|\chi(t)\|^2 + \frac{1}{2}(\kappa\chi(t), \chi(t)) + (W(\chi(t)), 1).$$

Recalling (2.3), (J1) and the fact that $|\chi| \le 1$ almost everywhere in $\Omega \times (0, T)$, for any given $T > 0$, it is not difficult to choose $\xi$ and $\eta$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + c_1\mathcal{E} + c_2\left(\|A\vartheta\|^2 + \|\chi_t\|^2\right) \le c_3 + c_4\|f\|^2, \qquad (3.2)$$

where $c_i$, $i = 1, \ldots, 4$, are positive constants. In particular, $c_1$ and $c_2$ depend on $\alpha$, while $c_3$ depends on $\Omega$, $\lambda$, $W(0)$ and $k_1$. Here we have also used the Young and Poincaré inequalities.

We now argue as in [35, Proof of Thm.2.1] and we test (2.6) by $\chi_t + \sigma\chi$, for some $\sigma > 0$ to be properly selected, but we do not integrate over $\Omega$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} + (\chi_t)^2 + \sigma\kappa\chi^2 + \sigma W'(\chi)\chi = (\mathcal{J}[\chi] + \alpha\vartheta)(\chi_t + \sigma\chi), \qquad (3.3)$$

where

$$\mathcal{G}(x,t) = \frac{\sigma}{2}(\chi(x,t))^2 + \frac{1}{2}\kappa(x)(\chi(x,t))^2 + W(\chi(x,t)), \qquad \text{a.e. in } \Omega, \; t \geq 0.$$

Reasoning as before and using Young's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(x,\cdot) + c_5 \mathcal{G}(x,\cdot) + c_6(\chi_t(x,\cdot))^2 \leq c_7(1 + |\vartheta(x,\cdot)|^2), \qquad \text{for a.a. } x \in \Omega,$$

where $c_i$, $i = 5, \ldots, 7$ are positive constants depending at most on $\Omega$, $\lambda$, $W(0)$, $k_1$ and $\alpha$. Thanks to the continuous embedding $D(A) \subset L^\infty(\Omega)$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(x,\cdot) + c_5 \mathcal{G}(x,\cdot) + c_6(\chi_t(x,\cdot))^2 \leq c_8(1 + \|A\vartheta\|^2), \qquad \text{for a.a. } x \in \Omega. \qquad (3.4)$$

If we multiply the above equation by $c_9 = \frac{c_2}{2c_8}$ and we add it to (3.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E} + c_9 \mathcal{G}(x,\cdot)) + c_1 \mathcal{E} + c_9 \mathcal{G}(x,\cdot) + \frac{c_2}{2}\|A\vartheta\|^2 + c_{10}\|\chi_t\|^2 \leq c_{11}(1 + \|f\|^2), \qquad (3.5)$$

for almost any $x \in \Omega$.

Applying now Gronwall's inequality to (3.5) we get, for all $t \geq 0$,

$$\mathcal{E}(t) + c_9 \mathcal{G}(x,t) \leq (\mathcal{E}(0) + c_9 \mathcal{G}(x,0))\, e^{-\mu t} + \frac{2c_{11}}{\mu_1}\left(1 + \|f\|^2\right), \qquad \text{for a.a. } x \in \Omega,$$

where $\mu_1 = \min\{1, c_1\}$, which yields (3.1). ∎

## 4  Existence of the global attractor

A consequence of inequality (3.1) is that $\mathcal{B}(R) = \{u \in V : \|u\| \leq R\}$ for a fixed $R > \sqrt{C_1}$ is absorbing for $\vartheta(t)$ as well as for $W(\chi(t))$. Therefore, consider, for instance,

$$X = \mathcal{B}(R) \times \{v \in L^\infty(\Omega) : |v| \leq 1 - \delta_1, \; \text{a.e. in } \Omega\}, \qquad (4.1)$$

where $\delta_1 \in (0,1)$ is such that

$$\{r \in (-1,1) : W(r) \leq R^2\} \subseteq [-1 + \delta_1, 1 + \delta_1].$$

If, for each $(\vartheta_0, \chi_0) \in X$ and any $t \geq 0$, we define

$$S(t)(\vartheta_0, \chi_0) = (\vartheta(t), \chi(t)),$$

then, thanks to (3.1), there exists $t_0 > 0$ such that $S(t)X \subseteq X$ for all $t \geq t_0$. If we endow $X$ with the $V \times H$-metric then we obtain a complete (bounded) metric space and $S(t)$ is strongly (Lipschitz) continuous semigroup on $X$ owing to (2.13) and (2.14). We now prove the main result of this note, namely, the dynamical system $(X, S(t))$ has a finite-dimensional global attractor.

**Theorem 4.1.** *Let the assumptions of Theorem 2.1 hold. In addition, suppose that*

$$\lambda_0 := \operatorname*{ess\,inf}_{x \in \Omega} \kappa(x) - \lambda > 0. \tag{4.2}$$

*Then $(X, S(t))$ possesses a finite-dimensional connected global attractor.*

PROOF. Consider $(\vartheta_{0i}, \chi_{0i}) \in X$, $i = 1, 2$, set

$$(\vartheta(t), \chi(t)) = ((\vartheta_1 - \vartheta_2)(t), (\chi_1 - \chi_2)(t))$$

where $(\vartheta_i(t), \chi_i(t)) = S(t)(\vartheta_{0i}, \chi_{0i})$ for $t \geq t_0$, and observe that

$$\vartheta_t + \alpha \chi_t + A\vartheta = 0, \qquad \text{a.e. in } \Omega \times (t_0, +\infty), \tag{4.3}$$

$$\chi_t + \kappa \chi + W'(\chi_1(t)) - W'(\chi_2(t)) = \mathcal{J}[\chi] + \alpha \vartheta, \qquad \text{a.e. in } \Omega \times (t_0, +\infty). \tag{4.4}$$

Let us multiply equation (4.3) by $A\vartheta(t)$. Integrating over $\Omega$, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \vartheta\|^2 + \|A\vartheta\|^2 = -(\alpha \chi_t, A\vartheta).$$

from which, using the Young and Poincaré inequalities, we derive the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \vartheta\|^2 + c \|\nabla \vartheta\|^2 \leq c_\alpha \|\chi_t\|^2.$$

and, by comparison in (4.4), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \vartheta\|^2 + c \|\nabla \vartheta\|^2 \leq c(k_1, \delta_1, \alpha) \left( \|\chi\|^2 + \|\mathcal{J}[\chi]\|^2 + \|\vartheta\|^2 \right). \tag{4.5}$$

for all $t \geq t_0$. On the other hand, multiplying (4.4) by $\chi(t)$ and integrating over $\Omega$ we find

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\chi\|^2 + (\kappa \chi, \chi) + (W'(\chi_1) - W'(\chi_2), \chi) = (\mathcal{J}[\chi], \chi) + \alpha(\vartheta, \chi),$$

and (4.2) entails

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\chi\|^2 + \lambda_0 \|\chi\|^2 \leq (\mathcal{J}[\chi], \chi) + \alpha(\vartheta, \chi).$$

Then, Young's inequality gives

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\chi\|^2 + \frac{\lambda_0}{2} \|\chi\|^2 \leq c_{\lambda_0} \|\mathcal{J}[\chi]\|^2 + c_\alpha \|\vartheta\|^2.$$

Since $\mathcal{J}$ is compact and self-adjoint we can find a finite-dimensional projector $\Pi_{\lambda_0}$ such that

$$\|\mathcal{J}[v]\|^2 \leq \frac{\lambda_0}{4c_{\lambda_0}} \|v\|^2 + \|\Pi_{\lambda_0}[v]\|^2, \tag{4.6}$$

for all $v \in H$. As a consequence we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\chi\|^2 + \frac{\lambda_0}{4} \|\chi\|^2 \leq c_{\lambda_0} \|\Pi[\chi]\|^2 + c_\alpha \|\vartheta\|^2. \tag{4.7}$$

Adding inequality(4.5) multiplied by $\mu_2 = \frac{\lambda_0}{8c(k_1,\delta_1,\alpha)}$ to (4.7) and using (4.6) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_2\|\nabla\vartheta\|^2 + \|\chi\|^2\right) + c\mu_2\|\nabla\vartheta\|^2 + \frac{\lambda_0}{8}\|\chi\|^2 \le c(\lambda_0, k_1, \delta_1, \alpha)\left(\|\Pi[\chi]\|^2 + \|\vartheta\|^2\right). \quad (4.8)$$

Therefore, from (4.8), we deduce

$$\|\vartheta(t)\|_V^2 + \|\chi(t)\|^2 \le c(\lambda_0, k_1, \delta_1, \alpha)e^{-\mu_3(t-t_0)}\left(\|\vartheta(t_0)\|_V^2 + \|\chi(t_0)\|^2\right) \quad (4.9)$$
$$+ c(\lambda_0, k_1, \delta_1, \alpha)\int_{t_0}^t\left(\|\vartheta(\tau)\|^2 + \|\Pi_{\lambda_0}[\chi(\tau)]\|^2\right)d\tau,$$

for all $t \in [t_0, T]$ and any fixed $T > t_0$. Here $\mu_3$ is a positive constant depending on $\lambda_0$, $k_1$, $\delta_1$, $\alpha$.

We now introduce the following pseudometric in $X$

$$\mathbf{d}_T((\vartheta_{01}, \chi_{01}), (\vartheta_{02}, \chi_{02})) = \left(\int_{t_0}^T\left(\|(\vartheta_1 - \vartheta_2)(\tau)\|^2 + \|\Pi_{\lambda_0}[(\chi_1 - \chi_2)(\tau)]\|^2\right)d\tau\right)^{1/2}$$

and we recall that a pseudometric is (pre)compact in $X$ (with respect to the topology induced by the $X$-metric) if any bounded sequence in $X$ contains a Cauchy subsequence with respect to $\mathbf{d}_T$ (see, for instance, [41, Def. 2.57]).

Let us prove that $\mathbf{d}_T$ is precompact in $X$. Let $\{(\vartheta_{0n}, \chi_{0n})\} \subset X$ ($X$ is bounded) and set $(\vartheta_n(t), \chi_n(t)) = S(t)(\vartheta_{0n}, \chi_{0n})$. Thanks to (2.11), we have that $\{\vartheta_n\}$ is bounded in $L^2(t_0, T; D(A)) \cap H^1(t_0, T; H)$. Therefore it contains a subsequence which strongly converges in $L^2(t_0, T, V)$. On the other hand, we have that $\{\Pi_{\lambda_0}[\chi_n]\}$ is bounded in $L^\infty(t_0, T; H)$. Also, by comparison in (2.6), we deduce that $\{(\chi_n)_t\}$ is bounded in $L^\infty(t_0, T; H)$. Therefore $\{(\Pi_{\lambda_0}[\chi_n])_t\}$ is bounded in $L^\infty(t_0, T; H)$ as well. Then $\{\Pi_{\lambda_0}[\chi_n(\cdot)]\}$ contains a subsequence which strongly converges in $L^2(t_0, T; H)$. Summing up $\{(\vartheta_{0n}, \chi_{0n})\}$ contains a Cauchy subsequence with respect to $\mathbf{d}_T$.

From (4.9), we deduce that there exists $t^* > t_0$ such that

$$\|S(t^*)(\vartheta_{01}, \chi_{01}) - S(t^*)(\vartheta_{02}, \chi_{02})\|_X$$
$$\le \frac{1}{2}\|(\vartheta_{01} - \vartheta_{02}, \chi_{01} - \chi_{02})\|_X + C(\lambda_0, k_1, \delta_1, \alpha)\mathbf{d}_{t^*}((\vartheta_{01}, \chi_{01}), (\vartheta_{02}, \chi_{02})).$$

Hence $S(t)$ has a (connected) global attractor (see [41, Thm. 2.56, Prop. 2.59]) of finite fractal (i.e., box counting) dimension (cfr. [37, Thm. 2.8.1]). ∎

**Remark 4.2.** Assumption (4.2) seems unavoidable when one wants to investigate the long-time behavior of solutions (cf. [8, (A4)] and [26, (1.19)]). In particular, thanks to this assumption, if we take $f \equiv 0$ and suppose $W$ real analytic then, on account of [35, Thm. 2.6], we have that the $\omega$-limit set of any pair $(\theta_0, \chi_0)$ satisfying (2.8) and (2.10) reduces to a singleton $\{(0, \chi_\infty)\}$, where

$$\kappa\chi_\infty + W'(\chi_\infty) = \mathcal{J}[\chi_\infty], \qquad \text{a.e. in }\Omega.$$

**Remark 4.3.** On account of Remark 2.5, we could take a larger phase space by replacing $V$ with $V_\rho = D(A^{\frac{\rho}{2}})$, $\rho \in (\frac{3}{4}, 1)$, in the definition of $\mathcal{B}(R)$ and endowing $X$ with $V_\rho \times H$-norm.

**Remark 4.4.** Consider the following equation

$$\chi_t + \kappa\chi + W'(\chi) = \mathcal{J}[\chi] + g,$$

where $g \in L^\infty(\Omega \times (0, +\infty))$. In this case it is possible to show a (uniform) separation property even when $W$ is a more general potential like, e.g., (1.3). However, one should use a comparison argument like, e.g., in [34]. Indeed, it is no longer sufficient to show the global uniform boundedness of $W(\chi)$ (cf. (3.1)). Then one can define a phase space given by the second component of $X$ (see (4.1)) and, arguing as above for the $\chi$ component only, prove the existence of a finite-dimensional global attractor.

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